

On $Sp(2M)$ Invariant Green Functions

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Abstract

Explicit form of two-point and three-point $Sp(2M)$ invariant Green functions is found.

1 Introduction

As was originally pointed out by Fronsdal [1], infinite towers of massless unitary representations of all spins of the AdS_4 symmetry $sp(4) \sim o(3,2)$ exhibit higher symplectic symmetry $sp(8)$ which extends usual conformal symmetry $su(2,2)$. It was also pointed out in [1] that the minimal space-time in which $sp(8)$ acts geometrically is ten dimensional. It can be realized as the space of Lagrangian planes [1] as well as the coset space $Sp(8)/P$ where P is an appropriate parabolic subgroup of $Sp(8)$ [2]. The local coordinates of this generalized space-time \mathcal{M}_4 are symmetric matrices $X^{\alpha\beta} = X^{\beta\alpha}$, where $\alpha, \beta \dots = 1, 2, 3, 4$ are four dimensional Majorana spinor indices. In this paper we will not distinguish between \mathcal{M}_4 and its large cell $R^{\frac{1}{2}M(M+1)}$. All integer spin massless bosons and half-integer massless fermions in four dimensions are described by a single scalar field $\Phi(X)$ and spinor field $\Phi_\alpha(X)$ in \mathcal{M}_4 , respectively.

Infinitesimal $Sp(8)$ transformations are [2]

$$\delta\Phi^d(X) = \left(\varepsilon^{\alpha\beta} \frac{\partial}{\partial X^{\alpha\beta}} + d\varepsilon^\alpha{}_\alpha + 2\varepsilon^\alpha{}_\beta X^{\beta\gamma} \frac{\partial}{\partial X^{\alpha\gamma}} - \varepsilon_{\alpha\beta} \left[dX^{\alpha\beta} + X^{\alpha\gamma} X^{\beta\eta} \frac{\partial}{\partial X^{\gamma\eta}} \right] \right) \Phi^d(X), \quad (1)$$

$$\delta\Psi_\rho^\Delta(X) = \left(\varepsilon^{\alpha\beta} \frac{\partial}{\partial X^{\alpha\beta}} + \Delta \varepsilon^\alpha{}_\alpha + 2\varepsilon^\alpha{}_\beta X^{\beta\gamma} \frac{\partial}{\partial X^{\alpha\gamma}} - \varepsilon_{\alpha\beta} \left[\Delta X^{\alpha\beta} + X^{\alpha\gamma} X^{\beta\eta} \frac{\partial}{\partial X^{\gamma\eta}} \right] \right) \Psi_\rho^\Delta(X) + (\varepsilon^\beta{}_\rho - \varepsilon_{\eta\rho} X^{\eta\beta}) \Psi_\beta^\Delta(X), \quad (2)$$

where $\varepsilon^{\alpha\beta}$, $\varepsilon^\alpha{}_\beta$, and $\varepsilon_{\alpha\beta}$ are X - independent parameters of, respectively, generalized translations, Lorentz transformations along with dilatations, and special conformal transformations. d and Δ are conformal weights of the fields Φ^d and Ψ_ρ^Δ , respectively.

The $sp(8)$ invariant form of the free massless field equations in \mathcal{M}_4 is [3]

$$\left(\frac{\partial^2}{\partial X^{\alpha\beta} \partial X^{\gamma\delta}} - \frac{\partial^2}{\partial X^{\alpha\gamma} \partial X^{\beta\delta}} \right) \Phi(X) = 0 \quad (3)$$

and

$$\frac{\partial}{\partial X^{\alpha\beta}} \Phi_\gamma(X) - \frac{\partial}{\partial X^{\alpha\gamma}} \Phi_\beta(X) = 0. \quad (4)$$

These equations are invariant under the $sp(8)$ transformations 1 and 2 with the canonical conformal dimensions $d = \Delta = \frac{1}{2}$. That these equations indeed describe four dimensional higher spin dynamics and are invariant under $sp(8)$ symmetry was shown in [3] based on the so called unfolded form of the higher spin equations in the form of certain covariant constancy conditions. Note that in [4] a $sp(8)$ world-line particle model was considered giving rise to the first-class constraints which upon Dirac quantization give rise to the equations analogous to those of [3]. The equations 3 and 4 make sense for any even number M of values taken by the indices α, β and remain $sp(2M)$ invariant. As argued in [4, 2], the related dynamical systems are expected to be related with conformal higher spin systems in six and ten dimensions for $M = 8$ and 16, respectively. We therefore consider in this paper any even M . Note that the form of the transformation laws 1 and 2 is M -independent.

It is yet an open question whether some nontrivial higher spin theories exist which exhibit higher symplectic symmetries in the unbroken phase beyond the free field level. There is some evidence, however, that spontaneously broken symmetries of this type play a role in supergravity [5]. Also spontaneously broken higher symplectic symmetries appear in the nonlinear higher spin field equations as formulated in [8]. If there is a phase in which higher symplectic symmetries are unbroken it should describe some higher spin theory because irreducible representations of higher symplectic symmetries form infinite towers of massless fields.

The aim of this note is to analyze general restrictions on $sp(2M)$ invariant Green functions. Following standard methods of conformal field theory along the lines of [6] (see also [7] and references therein) we derive the forms of two-point and three-point functions invariant under the $Sp(2M)$ transformations 1 and 2. The final results are

$$\langle \Phi^{d_1}(X_1) \Phi^{d_2}(X_2) \rangle = C_{\Phi\Phi} (\det |X_1 - X_2|)^{-d_1}, \quad d_1 = d_2 \quad (5)$$

$$\langle \Psi_{\rho_1}^{\Delta_1}(X_1) \Psi_{\rho_2}^{\Delta_2}(X_2) \rangle = C_{\Psi\Psi}(X_1 - X_2)_{\rho_1\rho_2} (\det |X_1 - X_2|)^{-\Delta_1}, \quad \Delta_1 = \Delta_2 \quad (6)$$

$$\begin{aligned} \langle \Phi^{d_1}(X_1) \Phi^{d_2}(X_2) \Phi^{d_3}(X_3) \rangle &= C_{\Phi\Phi\Phi} (\det |X_1 - X_3|)^{-\frac{1}{2}(d_1+d_3-d_2)} \\ &\times (\det |X_2 - X_3|)^{-\frac{1}{2}(d_2+d_3-d_1)} (\det |X_1 - X_2|)^{-\frac{1}{2}(d_1+d_2-d_3)}, \end{aligned} \quad (7)$$

$$\begin{aligned} \langle \Psi_{\rho_1}^{\Delta_1}(X_1) \Psi_{\rho_2}^{\Delta_2}(X_2) \Phi^d(X_3) \rangle &= C_{\Psi\Psi\Phi}(X_1 - X_2)_{\rho_1\rho_2} (\det |X_1 - X_3|)^{-\frac{\Delta_1+\Delta_2+d}{2}} \\ &\times (\det |X_2 - X_3|)^{-\frac{\Delta_2-\Delta_1+d}{2}} (\det |X_1 - X_2|)^{-\frac{\Delta_1+\Delta_2-d}{2}}. \end{aligned} \quad (8)$$

(Green functions containing odd numbers of fermions Ψ vanish.)

This form of $sp(2M)$ invariant Green functions is analogous to that of usual conformal Green functions with the invariant intervals replaced by the determinants of matrix coordinates. It is tempting to speculate that the obtained results should admit an extension to higher Green functions and/or multispinor fields. The fact that $sp(2M)$ invariance allows nonzero three-point Green functions we interpret as an indication that nontrivial generalized conformal theories with higher symplectic symmetries may exist beyond the free field level.

In the rest of this letter we sketch the proof of the formulae 5-8.

2 Two-point functions

As a consequence of 1 the $sp(2M)$ two point function

$$G(X_1, X_2) = \langle \Phi^{d_1}(X_1) \Phi^{d_2}(X_2) \rangle$$

has to satisfy the following conditions

$$\varepsilon^{\alpha\beta} \left\{ \frac{\partial}{\partial X_1^{\alpha\beta}} + \frac{\partial}{\partial X_2^{\alpha\beta}} \right\} G(X_1, X_2) = 0, \quad (9)$$

$$\left\{ \varepsilon^\alpha{}_\alpha (d_1 + d_2) + 2\varepsilon^\alpha{}_\beta \left(X_1^{\beta\gamma} \frac{\partial}{\partial X_1^{\alpha\gamma}} + X_2^{\beta\gamma} \frac{\partial}{\partial X_2^{\alpha\gamma}} \right) \right\} G(X_1, X_2) = 0, \quad (10)$$

$$\varepsilon_{\alpha\beta} \left\{ d_1 X_1^{\alpha\beta} + X_1^{\alpha\gamma} X_1^{\beta\eta} \frac{\partial}{\partial X_1^{\gamma\eta}} + d_2 X_2^{\alpha\beta} + X_2^{\alpha\gamma} X_2^{\beta\eta} \frac{\partial}{\partial X_2^{\gamma\eta}} \right\} G(X_1, X_2) = 0. \quad (11)$$

From 9 it follows that

$$G(X_1, X_2) = F(Y), \quad Y = X_1 - X_2 \quad (12)$$

Eq. 10 then transforms to

$$\varepsilon^\alpha{}_\alpha (d_1 + d_2) + 2\varepsilon^\alpha_\beta Y^{\beta\gamma} \frac{\partial}{\partial Y^{\alpha\gamma}} P(Y) = 0, \quad (13)$$

where $P(Y) = \ln F(Y)$. Taking into account

$$\frac{\partial}{\partial X^{\alpha\beta}} \det |X| = X_{\alpha\beta} \det |X|, \quad X_{\beta\gamma} X^{\gamma\alpha} = \delta_\beta^\alpha, \quad (14)$$

the general solution of this equation is

$$P(Y) = -\frac{d_1 + d_2}{2} \ln(\det |Y|) + C, \quad (15)$$

i.e.,

$$G(X_1, X_2) = C(\det |X_1 - X_2|)^{-\frac{d_1 + d_2}{2}}, \quad (16)$$

where C is an arbitrary constant. Substituting 16 into 11 one gets

$$(d_1 - d_2) \epsilon^{\alpha\beta} (X_1 - X_2)_{\alpha\beta} = 0, \quad (17)$$

i.e. $G(X_1, X_2) = \langle \Phi^{d_1}(X_1) \Phi^{d_2}(X_2) \rangle$ can be nonzero only if $d_1 = d_2$. As a result, we get the scalar $sp(2M)$ invariant two-point function 5. For the free field case of $d_1 = d_2 = \frac{1}{2}$ this agrees with the free Green function found in [2].

Consider now the fermionic two point function

$$G_{\rho_1 \rho_2}(X_1, X_2) = \langle \Psi_{\rho_1}^{\Delta_1}(X_1) \Psi_{\rho_2}^{\Delta_2}(X_2) \rangle.$$

Taking into account 2, the analysis analogous to the scalar case (cf. Eqs.12-17) gives $\Delta_1 = \Delta_2$ and

$$G_{\rho_1 \rho_2}(X_1, X_2) = (\det |Y|)^{-\Delta_1} P_{\rho_1 \rho_2}(Y), \quad (18)$$

where $P_{\rho_1 \rho_2}(Y)$ is some function satisfying the equations

$$2Y^{\beta\eta} \frac{\partial}{\partial Y^{\alpha\eta}} P_{\rho_1 \rho_2}(Y) + \delta^\beta_{\rho_1} P_{\alpha \rho_2}(Y) + \delta^\beta_{\rho_2} P_{\rho_1 \alpha}(Y) = 0, \quad (19)$$

$$2Y^{\alpha\delta} Y^{\beta\eta} \frac{\partial}{\partial Y^{\delta\eta}} P_{\rho_1 \rho_2}(Y) + \delta^\alpha_{\rho_1} Y^{\beta\delta} P_{\delta \rho_2}(Y) + \delta^\beta_{\rho_1} Y^{\alpha\delta} P_{\delta \rho_2}(Y) = 0. \quad (20)$$

From 19, 20 it follows that.

$$\delta^\alpha_{\rho_1} Y^{\beta\delta} P_{\delta \rho_2}(Y) - \delta^\beta_{\rho_2} Y^{\alpha\delta} P_{\rho_1 \delta}(Y) = 0$$

and, therefore,

$$P_{\rho_1 \rho_2}(Y) = (Y)_{\rho_1 \rho_2} P(Y). \quad (21)$$

Plugging 21 into 19 and 20 we find that $P(Y) = \text{const.}$ Finally we obtain the expression 6 for the fermionic Green function. For the canonical dimensions $\Delta_1 = \Delta_2 = \frac{1}{2}$ one recovers the free fermionic Green function of [2].

3 Three-point functions

As a consequence of 1 the Green function

$$G(X_1, X_2, X_3) = \langle \Phi^{d_1}(X_1) \Phi^{d_2}(X_2) \Phi^{d_3}(X_3) \rangle$$

has to satisfy the following system of equations

$$\varepsilon^{\alpha\beta} \sum_{i=1}^3 \frac{\partial}{\partial X_i^{\alpha\beta}} G(X_1, X_2, X_3) = 0, \quad (22)$$

$$\left\{ \varepsilon_\alpha^\alpha \sum_{i=1}^3 d_i + 2\varepsilon_\beta^\alpha \sum_{i=1}^3 X_i^{\beta\gamma} \frac{\partial}{\partial X_i^{\alpha\gamma}} \right\} G(X_1, X_2, X_3) = 0, \quad (23)$$

$$\varepsilon_{\alpha\beta} \sum_{i=1}^3 \left\{ d_i X_i^{\alpha\beta} + X_i^{\alpha\gamma} X_i^{\beta\eta} \frac{\partial}{\partial X_i^{\gamma\eta}} \right\} G(X_1, X_2, X_3) = 0. \quad (24)$$

From 22 we obtain that

$$G(X_1, X_2, X_3) = F(Y_1, Y_2), \quad (25)$$

where we introduced new variables:

$$Y_1^{\alpha\beta} = X_1^{\alpha\beta} - X_3^{\alpha\beta}, \quad Y_2^{\alpha\beta} = X_2^{\alpha\beta} - X_3^{\alpha\beta}. \quad (26)$$

Eq.23 then transforms to

$$\varepsilon_\alpha^\alpha \sum_{i=1}^3 d_i + 2\varepsilon_\beta^\alpha \sum_{i=1}^2 Y_i^{\beta\gamma} \frac{\partial}{\partial Y_i^{\alpha\gamma}} P(Y_1, Y_2) = 0, \quad (27)$$

where $P(Y_1, Y_2) = \ln F(Y_1, Y_2)$. Its general solution is

$$P(Y_1, Y_2) = \ln \left[(\det |Y_1|)^{-\frac{k_1}{2}} (\det |Y_2|)^{-\frac{k_2}{2}} (\det |Y_1 - Y_2|)^{-\frac{k_3}{2}} \right] + R(Y_1, Y_2), \quad (28)$$

where k_1, k_2 and k_3 are such that

$$k_1 + k_2 + k_3 = d_1 + d_2 + d_3 \quad (29)$$

and $R(Y_1, Y_2)$ satisfies the equation

$$\left\{ Y_1^{\beta\gamma} \frac{\partial}{\partial Y_1^{\alpha\gamma}} + Y_2^{\beta\gamma} \frac{\partial}{\partial Y_2^{\alpha\gamma}} \right\} R(Y_1, Y_2) = 0 \quad (30)$$

Taking into account 25 and 27 eq.24 transforms to

$$d_1 Y_1^{\alpha\beta} + d_2 Y_2^{\alpha\beta} + \left[Y_1^{\alpha\gamma} Y_1^{\beta\eta} \frac{\partial}{\partial Y_1^{\gamma\eta}} + Y_2^{\alpha\gamma} Y_2^{\beta\eta} \frac{\partial}{\partial Y_2^{\gamma\eta}} \right] P(Y_1, Y_2) = 0 \quad (31)$$

Plugging here the expression 28 and choosing

$$2d_1 - k_1 - k_3 = 0, \quad 2d_1 - k_1 - k_3 = 0, \quad (32)$$

we get

$$\left[Y_1^{\alpha\gamma} Y_1^{\beta\eta} \frac{\partial}{\partial Y_1^{\gamma\eta}} + Y_2^{\alpha\gamma} Y_2^{\beta\eta} \frac{\partial}{\partial Y_2^{\gamma\eta}} \right] R(Y_1, Y_2) = 0. \quad (33)$$

It is easy to see that from 30 and 33 it follows $R(Y_1, Y_2) = \text{const}$. Thus 7 is the general expression for a three-point function.

For the three-point function

$$G_{\rho_1\rho_2}(X_1, X_2, X_3) = \langle \Psi_{\rho_1}^{\Delta_1}(X_1) \Psi_{\rho_2}^{\Delta_2}(X_2) \Phi^d(X_3) \rangle$$

one obtains analogously from 1 and 2 the expression 8. Also, one can see that the Green functions with odd numbers of spinor fields vanish.

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